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A REPRESENTATION FOR SELF-SIMILAR PROCESSES

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A self-similar process $Z(t)$ has stationary increments and is invariant in law under the transformation $Z(t) \to c^{-H}Z(ct)$, $c \geq 0$. The choice $\frac{1}{2} < H < 1$ ensures that the increments of $Z(t)$ exhibit a long range positive correlation.

Mandelbrot and Van Ness investigated the case where $Z(t)$ is Gaussian and represented that Gaussian self-similar process as a fractional integral of Brownian motion. They called it fractional Brownian motion. This paper provides a time-indexed representation for a sequence of self-similar processes $\bar{Z}_m(t)$, $m = 1, 2, \ldots$, whose finite-dimensional moments have been specified in an earlier paper. $\bar{Z}_1(t)$ is the Gaussian fractional Brownian motion but the processes $\bar{Z}_m(t)$ are not Gaussian when $m \geq 2$.

Self-similar processes are being studied in physics, in the context of the renormalization group theory for critical phenomena, and in hydrology where they account for the so-called "Hurst effect".

self-similar processes	renormalization group theory
multiple stochastic integrals	hydrology
Hermite polynomials	Hurst effect
fractional Brownian motion	

1. Introduction

Let $\frac{1}{2} < H < 1$. A process $\{Z(t), t \geq 0\}$ is *self-similar with parameter H* if it has stationary increments, satisfies $Z(0) = 0$, and is invariant (in law) under the transformation (self-similarity)

$$Z(t) \to c^{-H}Z(ct), \quad c \geq 0. \tag{1.1}$$

We are interested here in self-similar processes that have finite variance (this restriction excludes the stable processes). All finite variance self-similar processes, whether Gaussian or not, have a covariance

$$\mathbf{E}\, Z(t_1)Z(t_2) = \tfrac{1}{2}\{t_1^{2H} + t_2^{2H} - |t_1 - t_2|^{2H}\}\mathbf{E}\, Z^2(1). \tag{1.2}$$

Mandelbrot and Van Ness [8] investigated the case where the self-similar process $Z(t)$ is Gaussian and represented that process as a fractional integral of Brownian motion. They called it *fractional Brownian motion*. Fractional Brownian motion plays an important role in hydrology, where it accounts for the so-called "Hurst effect". There is a need, however, for non-Gaussian self-similar processes (see Lawrance and Kottegoda [5]).

The purpose of this paper is to establish a time-indexed representation for a sequence of self-similar processes $\bar{Z}_m(t)$, $m = 1, 2, \ldots$, whose finite-dimensional moments have been specified in Taqqu [13]. $\bar{Z}_1(t)$ is the Gaussian fractional Brownian motion but the processes $\bar{Z}_m(t)$ are not Gaussian when $m \geq 2$. We show here that each $\bar{Z}_m(t)$, $m = 1, 2, \ldots$, admits the following representation:

$$\begin{aligned}\bar{Z}_m(t) = C(m, D)\Bigg\{ &\int_{-\infty}^{0} \mathrm{d}B(\xi_1) \int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2) \cdots \int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m) \int_0^t \prod_{i=1}^{m} (s-\xi_i)^{-D/2-1/2}\, \mathrm{d}s \\ &+ \int_0^t \mathrm{d}B(\xi_1) \int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2) \cdots \int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m) \\ &\cdot \int_{\xi_1}^{t} \prod_{i=1}^{m} (s-\xi_i)^{-D/2-1/2}\, \mathrm{d}s \Bigg\} \end{aligned} \qquad (1.3)$$

where $0 < D < 1/m$, $B(\xi)$ is standard Brownian motion and $C(m, D)$ is a normalization constant. The self-similarity parameter is $H = 1 - \frac{1}{2}mD$. The multiple stochastic integrals are defined as in Itô [3]. The $\bar{Z}_m(t)$'s belong to $L^2(W)$, the space of square integrable functionals with respect to Wiener measure. A representation of these processes in the spectral domain was recently obtained by Dobrushin [2] using random generalized fields. When $m = 1$, the time-indexed representation obtained here is equivalent to that used by Mandelbrot and Van Ness [8] to define the Gaussian fractional Brownian motion.

Rosenblatt [11] first hinted at the possible existence of a non-Gaussian self-similar process by constructing an example of a stationary sequence of random variables that is not strongly mixing and whose normalized sum does not obey the usual central limit theorem. The weak limit of that sum turns out to have the marginal distribution of the "Rosenblatt process", a non-Gaussian self-similar process whose existence was established by Taqqu [12]. $\bar{Z}_2(t)$ is the Rosenblatt process.

Self-similar processes are of interest in physics, in the context of the renormalization group theory, and in hydrology, where they account for the so-called "Hurst effect". See Jona-Lasinio [4] for a review of the physics literature and Lawrence and Kottegoda [5] for a review of the literature in hydrology.

In hydrology, an adequate stochastic model should provide means of data generation. Such simulated data is used to study the responses of water resources systems and to analyze the practical efficiency of statistical estimation procedures.

The time-indexed representation of the Gaussian self-similar process introduced by Mandelbrot and Van Ness [8] has served such a purpose (see Mandelbrot and Wallis [9, 10] Mandelbrot [6] and Chi et al. [1]). The time indexed representation of the non-Gaussian self-similar processes obtained here can lead to data generation of finite variance, non-Gaussian time series that exhibit the Hurst effect. This may be achieved efficiently by using the results of this paper in conjunction with those of Taqqu [12, 13].

2. The Hermite moment condition

Introduce the Hermite polynomials

$$H_m(x)=(-1)^m \, \mathrm{e}^{x^2/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathrm{e}^{-x^2/2}, \quad m=0,1,2,\ldots .$$

Thus $H_1(x)=x$, $H_2(x)=x^2-1$.

Fix $m\geq 1$. The process $\{Z_m(t), t\geq 0\}$ is said to satisfy a *Hermite moment condition* if the following holds for all $p\geq 1$ and $0\leq t_1, t_2, \ldots, t_p<\infty$:

(a) $\mathbf{E}\, Z_m(t_1)\cdots Z_m(t_p)=0$ whenever $p=1$ or mp is odd;

(b) when $p\geq 2$ and mp is even, there exists a constant $K>0$ and a function $r(s_1,s_2)$, $0\leq s_1, s_2<\infty$, such that

$$\mathbf{E}\, Z_m(t_1)\cdots Z_m(t_p)=K^p\frac{(m!)^p}{2^{mp/2}(\frac{1}{2}mp!)}\sum\int_0^{t_1}\mathrm{d}s_1\cdots\int_0^{t_p}\mathrm{d}s_p\, r(s_{i_1},s_{j_1})\cdots r(s_{i_q},s_{j_q}), \tag{2.1}$$

where $q=\frac{1}{2}mp$ and Σ is a sum over all indices $i_1, j_1, \ldots, i_q, j_q$ satisfying

(i) $i_1, j_1, \ldots, i_q, j_q \in\{1,2,\ldots,p\}$

(ii) $i_1\neq j_1, \ldots, i_q\neq j_q$

(iii) each number $1,2,\ldots,p$ appears exactly m times in $(i_1, j_1, \ldots, i_q, j_q)$.

Remark 2.1. Taqqu [13, Theorem 3] specifies a sequence of finite-dimensional moments parameterized by $m=1,2,\ldots$. These are the moments of processes $\tilde{Z}_m(t)$, $m=1,2,\ldots$ that satisfy the Hermite moment condition with

$$r(s_1,s_2)=|s_1-s_2|^{-D}, \quad 0<D<1/m.$$

Our goal is to obtain a representation of $\tilde{Z}_m(t)$, $m=1,2,\ldots$. As a first step we obtain a representation of processes $Z_m(t)$ that satisfy a Hermite moment condition with an $r(s_1,s_2)<\infty$ for all $0\leq s_1, s_2<\infty$.

Let $B(t)$ denote standard Brownian motion. A functional F belongs to $L^2(W)$ if $\mathbf{E}\{F(B(\cdot))\}^2<\infty$.

Lemma 2.2. *Let $m \geq 1$ and let $e(s, \xi)$, $0 \leq s < \infty$, $-\infty < \xi < +\infty$, be a function that satisfies*

$$\int_{-\infty}^{+\infty} e^2(s, \xi)\, d\xi < \infty \tag{2.2}$$

for all $0 \leq s < \infty$. Then the process $\{Z_m(t), t \geq 0\}$ given by

$$Z_m(t) = \int_0^t ds \int_{-\infty}^{+\infty} dB(\xi_1) e(s, \xi_1) \int_{-\infty}^{\xi_1} dB(\xi_2) e(s, \xi_2) \times \int_{-\infty}^{\xi_2} dB(\xi_3) e(s, \xi_3) \cdots \int_{-\infty}^{\xi_{m-1}} dB(\xi_m) e(s, \xi_m) \tag{2.3}$$

is well-defined in $L^2(W)$ and satisfies the Hermite moment condition with $K = 1/m!$ and with

$$r(s_1, s_2) = \int_{-\infty}^{+\infty} e(s_1, \xi) e(s_2, \xi)\, d\xi < \infty.$$

Proof. $r(s_1, s_2) < \infty$ by (2.2). (2.2) also ensures that $Z_m(t)$ is well-defined in $L^2(W)$ and that it can be expressed as

$$Z_m(t) = \int_0^t ds \frac{1}{m!} (r(s, s))^{m/2} H_m(X(s))$$

where

$$X(s) = \frac{1}{(r(s, s))^{1/2}} \int_{-\infty}^{+\infty} e(s, \xi)\, dB(\xi)$$

(see for example, McKean [6]). Thus,

$$EZ_m(t_1) \cdots Z_m(t_p) = \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds_p \frac{(r(s, s))^{mp/2}}{(m!)^p} \mathbf{E}\, H_m(X(s_1)) \cdots H_m(X(s_p)).$$

By [13, Proposition 3.1], $X(s)$ satisfies a Hermite moment condition. In fact, for $p \geq 2$ and m_p even, its moments are

$$\mathbf{E}\, H_m(X(s_1)) \cdots H_m(X(s_p)) = \frac{(m!)^p}{2^{mp/2}(\frac{1}{2}mp!)} \sum R(s_{i_1}, s_{j_1}) \cdots R(s_{i_q}, s_{j_q}),$$

with $R(s_i, s_j) = r(s_i, s_j)/(r(s_i, s_i))^{1/2}(r(s_j, s_j))^{1/2}$, $q = \frac{1}{2}mp$ and $\sum$ defined as in (2.1). But each number $1, 2, \ldots$, p appears exactly m times in $(i_1, j_1, \ldots, i_q, j_q)$. Therefore

$$\mathbf{E}\, Z_m(t_1) \cdots Z_m(t_p) = \frac{1}{2^{mp/2}(\frac{1}{2}mp!)} \sum \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds_p\, r(s_{i_1}, s_{j_1}) \cdots r(s_{i_q}, s_{j_q}).$$

This concludes the proof.

The next lemma involves a weakened version of condition (2.2).

Lemma 2.3. *Fix $m \geq 1$, $\eta > 0$ and let $e(s, \xi, \varepsilon)$, $0 \leq s < \infty$, $-\infty < \xi < +\infty$, $0 < \varepsilon < \eta$, be non-negative functions satisfying*

$$\int_{-\infty}^{+\infty} e^2(s, \xi, \varepsilon)\, d\xi < \infty \tag{2.4}$$

for each s and ε. Suppose that $e(s, \xi, \varepsilon)$ increases monotonically to $e(s, \xi)$ as $\varepsilon \to 0$ for all $0 \leq s < \infty$ and $-\infty < \xi < +\infty$, and that for all $t > 0$

$$\int_0^t ds_1 \int_0^t ds_2 \left\{\int_{-\infty}^{+\infty} e(s_1, \xi) e(s_2, \xi)\, d\xi\right\}^m < \infty. \tag{2.5}$$

Then the process $\{Z_m(t), t \geq 0\}$ given by

$$Z_m(t) = \int_{-\infty}^{+\infty} dB(\xi_1) \int_{-\infty}^{\xi_1} dB(\xi_2) \cdots \int_{-\infty}^{\xi_{m-1}} dB(\xi_m) \int_0^t \prod_{i=1}^m e(s, \xi_i)\, ds \tag{2.6}$$

is well-defined in $L^2(W)$ and satisfies the Hermite moment condition with $K = 1/m!$ and with

$$r(s_1, s_2) = \int_{-\infty}^{+\infty} e(s_1, \xi) e(s_2, \xi)\, d\xi.$$

Proof. $Z_m(t)$ is well-defined in $L^2(W)$ because

$$\begin{aligned}\mathbf{E}\, Z_m^2(t) &= \int_{-\infty}^{+\infty} d\xi_1 \int_{-\infty}^{\xi_1} d\xi_2 \cdots \int_{-\infty}^{\xi_{m-1}} d\xi_m \int_0^t ds_1 \int_0^t ds_2 \prod_{i=1}^m e(s_1, \xi_i) e(s_2, \xi_i) \\ &\leq \int_0^t ds_1 \int_0^t ds_2 \left\{\int_{-\infty}^{+\infty} e(s_1, \xi) e(s_2, \xi)\, d\xi\right\}^m\end{aligned}$$

which is finite by (2.5).

Now let $Z_m(t, \varepsilon)$ be the $Z_m(t)$ of (2.3) with $e(s, \xi)$ replaced by $e(s, \xi, \varepsilon)$. (2.4) ensures that $Z_m(t, \varepsilon)$ is well defined in $L^2(W)$. Also,

$$\begin{aligned}&\mathbf{E}\, Z_m(t_1, \varepsilon) \cdots Z_m(t_p, \varepsilon) = \\ &= \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds_p \cdot \int_{-\infty}^{+\infty} d\xi_1^1 \cdots \int_{-\infty}^{+\infty} d\xi_1^p \cdot \int_{-\infty}^{\xi_1^1} d\xi_2^1 \cdots \int_{-\infty}^{\xi_1^p} d\xi_2^p \\ &\quad \cdot \int_{-\infty}^{\xi_2^1} d\xi_3^1 \cdots \int_{-\infty}^{\xi_2^p} d\xi_3^p \cdots \int_{-\infty}^{\xi_{m-1}^1} d\xi_m^1 \cdots \int_{-\infty}^{\xi_{m-1}^p} d\xi_m^p \\ &\quad \cdot \Gamma(\xi_i^j, i = 1, \ldots, m; j = 1, \ldots, p) \prod_{i=1}^m \prod_{j=1}^p e(s_j, \xi_i^j, \varepsilon)\end{aligned}$$

where Γ is a sum of products of delta functions involving the ξ_i^j but not involving ε.

The integrand of the multiple integral is non-negative and $e(s, \xi, \varepsilon)$ increases monotonically to $e(s, \xi)$ as $\varepsilon \to 0$. By the monotone convergence theorem

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \mathbf{E}\, Z_m(t_1, \varepsilon) \cdots Z_m(t_p, \varepsilon) \\
&= \int_{-\infty}^{+\infty} d\xi_1^1 \cdots \int_{-\infty}^{+\infty} d\xi_1^p \int_{-\infty}^{\xi_1^1} d\xi_2^1 \cdots \int_{-\infty}^{\xi_1^p} d\xi_2^p \\
&\quad \cdot \int_{-\infty}^{\xi_2^1} d\xi_3^1 \cdots \int_{-\infty}^{\xi_2^p} d\xi_3^p \cdots \int_{-\infty}^{\xi_{m-1}^1} d\xi_m^1 \cdots \int_{-\infty}^{\xi_{m-1}^p} d\xi_m^p \\
&\quad \cdot \Gamma(\xi_i^j, i = 1, \ldots, m; j = 1, \ldots, p) \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds_p \prod_{i=1}^{m} \prod_{j=1}^{p} e(s_j, \xi_i^j) \\
&= \mathbf{E}\, Z_m(t_1) \cdots Z_m(t_p).
\end{aligned}
$$

We now evaluate $\lim_{\varepsilon \to 0} \mathbf{E}\, Z_m(t_1, \varepsilon) \cdots Z_m(t_p, \varepsilon)$.

Lemma 2.2 applies to $Z_m(t, \varepsilon)$ because of condition (2.4). $Z_m(t, \varepsilon)$ satisfies then the Hermite moment condition with $K = 1/m!$ and

$$
r(s_i, s_j, \varepsilon) = \int_{-\infty}^{+\infty} e(s_i, \xi, \varepsilon) e(s_j, \xi, \varepsilon)\, d\xi.
$$

Therefore

$$
\lim_{\varepsilon \to 0} \mathbf{E}\, Z_m(t_1, \varepsilon) \cdots Z_m(t_p, \varepsilon) = 0
$$

whenever $p = 1$ or mp is odd. Suppose now $p \geq 2$ and mp even. Using the notation introduced in (2.1) we get

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \mathbf{E}\, Z_m(t_1, \varepsilon) \cdots Z_m(t_p, \varepsilon) \\
&= \frac{1}{2^{mp/2}(\frac{1}{2}mp!)} \sum \lim_{\varepsilon \to 0} \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds_p\, r(s_{i_1}, s_{j_1}, \varepsilon) \cdots r(s_{i_q}, s_{j_q}, \varepsilon) \\
&= \frac{1}{2^{mp/2}(\frac{1}{2}mp!)} \sum \int_0^{t_1} ds_1 \cdots \int_0^{t_p} ds\, r(s_{i_1}, s_{j_1}) \cdots r(s_{i_q}, s_{j_q})
\end{aligned}
$$

where

$$
\begin{aligned}
r(s_i, s_j) &= \lim_{\varepsilon \to 0} r(s_i, s_j, \varepsilon) \\
&= \int_{-\infty}^{+\infty} e(s_i, \xi) e(s_j, \xi)\, d\xi.
\end{aligned}
$$

The monotone convergence theorem justifies taking the limit under the integration signs. This completes the proof of the lemma.

3. Self-similar processes

Theorem 3.1. *Let $m \geq 1$, $0 < D < 1/m$ and*

$$C(m, D) = \left\{ \frac{(1-mD)(2-mD)}{2\left(\int_0^{\infty} (u+u^2)^{-D/2-1/2} \, \mathrm{d}u\right)^m} \right\}^{1/2}.$$

Then the process $\{\bar{Z}_m(t), t \geq 0\}$ given by

$$\bar{Z}_m(t) = C(m, D)\left\{ \int_{-\infty}^{0} \mathrm{d}B(\xi_1) \int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2) \cdots \int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m) \int_0^t \prod_{i=1}^{m} (s-\xi_i)^{-D/2-1/2} \, \mathrm{d}s \right.$$

$$+ \int_0^t \mathrm{d}B(\xi_1) \int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2) \cdots \int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m)$$

$$\left. \cdot \int_{\xi_1}^{t} \prod_{i=1}^{m} (s-\xi_i)^{-D/2-1/2} \, \mathrm{d}s \right\} \tag{3.1}$$

is well-defined in $L^2(W)$, is normalized ($\mathbf{E}\, \bar{Z}_m^2(1) = 1$) and satisfies the Hermite moment condition with

$$K = \frac{(1-mD)(2-mD)}{2(m!)}, \qquad r(s_1, s_2) = |s_1 - s_2|^{-D}.$$

The process is self-similar with parameter $H = 1 - mD/2$.

Proof. Let

$$e(s, \xi, \varepsilon) = \begin{cases} (s-\xi+\varepsilon)^{-D/2-1/2} & \text{if } \xi < s, \\ 0 & \text{otherwise.} \end{cases}$$

be defined for $-\infty < \xi < \infty$, $0 \leq s < \infty$, $\varepsilon > 0$. $e(s, \xi, \varepsilon)$ satisfies condition (2.4) of Lemma 2.3 because

$$\int_{-\infty}^{+\infty} e^2(s, \xi, \varepsilon) \, \mathrm{d}\xi = \int_{-\infty}^{s} (s-\xi+\varepsilon)^{-D-1} \, \mathrm{d}\xi = \frac{\varepsilon^{-D}}{D} < \infty.$$

Now, as $\varepsilon \to 0$, $e(s, \xi, \varepsilon)$ increases monotonically to

$$e(s, \xi) = \begin{cases} (s-\xi)^{-D/2-1/2} & \text{if } \xi < s, \\ 0 & \text{otherwise,} \end{cases}$$

and $e(s,\xi)$ satisfies condition (2.5) of Lemma 2.3. Indeed, for $s_1 \neq s_2$,

$$\int_{-\infty}^{+\infty} e(s_1,\xi)e(s_2,\xi)\,\mathrm{d}\xi = \int_{-\infty}^{s_1\wedge s_2} (s_1-\xi)^{-D/2-1/2}(s_2-\xi)^{-D/2-1/2}\,\mathrm{d}\xi$$

$$= \int_0^{\infty} u^{-D/2-1/2}(|s_1-s_2|+u)^{-D/2-1/2}\,\mathrm{d}u$$

$$= a|s_1-s_2|^{-D}$$

where

$$a = \int_0^{\infty} (u+u^2)^{-D/2-1/2}\,\mathrm{d}u < \infty.$$

Hence

$$\int_0^t \mathrm{d}s_1 \int_0^t \mathrm{d}s_2 \left\{\int_{-\infty}^{+\infty} e(s_1,\xi)e(s_2,\xi)\,\mathrm{d}\xi\right\}^m = a^m \int_0^t\int_0^t |s_1-s_2|^{-mD}\,\mathrm{d}s_1\,\mathrm{d}s_2$$

$$= \frac{1}{C^2(m,D)} < \infty.$$

Lemma 2.3 ensures that

$$\begin{aligned}
\tilde{Z}_m(t) &= C(m,D)\left\{\int_{-\infty}^{0} \mathrm{d}B(\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m)\int_0^t \prod_{i=1}^m e(s,\xi_i)\,\mathrm{d}s\right.\\
&\qquad \left.+\int_0^{\infty} \mathrm{d}B(\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m)\int_0^t \prod_{i=1}^m e(s,\xi_i)\,\mathrm{d}s\right\}\\
&= C(m,D)\left\{\int_{-\infty}^{0} \mathrm{d}B(\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m)\int_0^t \prod_{i=1}^m (s-\xi_i)^{-D/2-1/2}\,\mathrm{d}s\right.\\
&\qquad +\int_0^{t} \mathrm{d}B(\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(\xi_m)\\
&\qquad \left.\cdot\int_{\xi_1}^t \prod_{i=1}^m (s-\xi_i)^{-D/2-1/2}\,\mathrm{d}s\right\}
\end{aligned}$$

is well-defined in $L^2(W)$ and that it satisfies a Hermite moment condition with $K = C(m,D)/m!$ and $r(s_1,s_2) = a|s_1-s_2|^{-D}$, or equivalently, with $K = C(m,D)a^{m/2}/m!$ and $r(s_1,s_2) = |s_1-s_2|^{-D}$.

That $Z_m(t)$ is self-similar with parameter $H = 1-\frac{1}{2}mD$ is readily verified. Indeed, choose $a>0$ and define $Z_m(at)$ through (3.1). A change of variable yields

$$\begin{aligned}
Z_m(at) &= a^{m(-D/2-1/2)+1}C(m,D)\\
&\quad \cdot\left\{\int_{-\infty}^{0} \mathrm{d}B(a\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(a\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(a\xi_m)\int_0^t \prod_{i=1}^m (s-\xi_i)^{-D/2-1/2}\,\mathrm{d}s\right.\\
&\quad \left.+\int_0^{t} \mathrm{d}B(a\xi_1)\int_{-\infty}^{\xi_1} \mathrm{d}B(a\xi_2)\cdots\int_{-\infty}^{\xi_{m-1}} \mathrm{d}B(a\xi_m)\int_{\xi_1}^t \prod_{i=1}^m (s-\xi_i)^{-D/2-1/2}\,\mathrm{d}s\right\}.
\end{aligned}$$

The multiple integral can be obtained as an $L^2(W)$ limit of integrals that have simple functions as integrands (Ito [3]). Once evaluated, these integrals involve terms of the type $B(I_1)B(I_2)\cdots B(I_m)$ where $I_1, I_2, \ldots, I_m$ are measurable sets in $\mathbf{R}^1$ with finite Lebesgue measure λ, and where $(B(I_1), B(I_2), \ldots, B(I_m))$ is multivariate normal with $\mathbf{E}\,B(I_i)=0$ and $\mathbf{E}\,B(I_i)B(I_j)=\lambda(I_i \cap I_j)$. Let aI be the set I rescaled by a. Obviously,

$$\mathbf{E}\,B(aI_i)B(aI_j)=a\lambda(I_i \cap I_j).$$

$(B(aI_1), B(aI_2), \ldots, B(aI_m))$ has then the same distribution as $(a^{1/2}B(I_1), a^{1/2}B(I_2), \ldots, a^{1/2}B(I_m))$. Therefore

$$\begin{aligned} Z_m(at) &\stackrel{\Delta}{=} a^{m(-D/2-1/2)+1+m/2}Z_m(t) \\ &= a^{1-mD/2}Z_m(t) \end{aligned}$$

where Δ indicates equality of the finite dimensional distributions. This completes the proof of the theorem.

Remark 3.2. The parameters m, H and D are related through $H=1-\frac{1}{2}mD$. Thus $0<D<1/m$ is equivalent to $\frac{1}{2}<H<1$, and the exponents $-\frac{1}{2}D-\frac{1}{2}$ that appear in (3.1) may be expressed as $H/m-1/m-\frac{1}{2}$.

The moments of $\bar{Z}_m(t)$ are given by (2.1) with K and $r(s_1, s_2)$ defined as in Theorem 3.1. For a more specific evaluation of some of the moments, see Taqqu [13, p. 228]. Note that

$$\mathbf{E}\,\bar{Z}_m(t_1)\bar{Z}_m(t_2)=\tfrac{1}{2}\{t_1^{2H}+t_2^{2H}-|t_1-t_2|^{2H}\} \tag{3.2}$$

for all $m \geq 1$. In fact, the right hand side of (3.2) represents the covariance of any mean 0, finite variance, normalized process $Z(t)$ which is self-similar with parameter H. For such a process

$$\mathbf{E}\,Z^2(t)=t^{2H}\mathbf{E}\,Z^2(1)=t^{2H}$$

and, because of the stationarity of the increments and $Z(0)=0$,

$$\begin{aligned} \mathbf{E}\,Z(t_1)Z(t_2) &= \tfrac{1}{2}\{\mathbf{E}\,Z^2(t_1)+\mathbf{E}\,Z^2(t_2)-\mathbf{E}\,(Z(t_1)-Z(t_2))^2\} \\ &= \tfrac{1}{2}\{t_1^{2H}+t_2^{2H}-|t_1-t_2|^{2H}\}. \end{aligned}$$

The process $\{\bar{Z}_m(t), t\geq 0\}$ defined in Theorem 3.1 represents (up to a multiplicative constant) the Gaussian fractional Brownian motion in the case $m=1$ and the Rosenblatt process in the case $m=2$ (Taqqu [12, 13]).

When $m=1$, the expression (3.1) becomes

$$\begin{aligned} \bar{Z}_1(t) &= C(1,D)\left\{\int_{-\infty}^{0} \mathrm{d}B(\xi)\int_0^t (s-\xi)^{-D/2-1/2}\,\mathrm{d}s+\int_0^t \mathrm{d}B(\xi)\int_\xi^t (s-\xi)^{-D/2-1/2}\,\mathrm{d}s\right\} \\ &= \frac{C(1,D)}{-\frac{1}{2}D+\frac{1}{2}}\left\{\int_{-\infty}^{0} \mathrm{d}B(\xi)((t-\xi)^{-D/2+1/2}-(-\xi)^{-D/2+1/2})+\int_0^t \mathrm{d}B(\xi)(t-\xi)^{-D/2+1/2}\right\} \end{aligned}$$

with $-\frac{1}{2}D+\frac{1}{2}=H-\frac{1}{2}$. This last expression is the one used by Mandelbrot and Van Ness [8] to define fractional Brownian motion.

A judicious modification of the integrands in (3.1) can provide other self-similar processes. For example, one may replace the m exponents $-\frac{1}{2}D-\frac{1}{2}$ by $-\frac{1}{2}D_1-\frac{1}{2}$, $-\frac{1}{2}D_2-\frac{1}{2}, \ldots, \frac{1}{2}D_m-\frac{1}{2}$ respectively where $D_1, D_2, \ldots, D_m > 0$ and $\Sigma_{i=1}^{m} D_i < 1$. The resulting process is still defined in $L^2(W)$ and is self-similar with parameter $H = 1-\frac{1}{2}\Sigma_{i=1}^{m} D_i$. The characterization of all self-similar processes is still an open problem.

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